CONNECTED COMPONENTS OF THE STRATA OF THE MODULI SPACE OF MEROMORPHIC DIFFERENTIALS

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Abstract. In this paper, we study the translation surfaces corresponding to meromorphic differentials on compact Riemann surfaces. We compute the number of connected components of the corresponding strata of the moduli space. We show that in genus greater than or equal to two, one has up to three components with a similar description as the one of Kontsevich and Zorich for the moduli space of Abelian differentials. In genus one, one can obtain an arbitrarily large number of connected components that are easily distinghished by a simple topological invariant.

Contents

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1. Introduction

A nonzero holomorphic one-form (Abelian differential) on a compact Riemann surface naturally defines a flat metric with conical singularities on this surface. Geometry and dynamics on such flat surfaces, in relation to geometry and dynamics on the corresponding moduli space of Abelian differentials is a very rich topic and has been widely studied in the last 30 years. It is related to interval exchange transformations, billards in polygons, Teichmüller dynamics.

A noncompact translation surface corresponds to a a one form on a noncompact Riemann surface. The dynamics and geometry on some special cases of noncompact translation surfaces have been studied more recently. For instance, dynamics on $\mathbb{Z}^d$ covers of compact translation surfaces (see [6, 8, 3]), infinite square tiled surfaces (see [9]), or general noncompact surfaces (see [1, 2, 15]).

In this paper, we investigate the case of translation surfaces that come from meromorphic differentials defined on compact Riemann surfaces. In this case, we obtain infinite area surfaces, with "finite complexity". Dynamics of the geodesic flow on a generic direction on such surface is trivial any infinite orbit converges to the poles. Also, $SL_2(\mathbb{R})$ action doesn't seem as rich as in the Abelian case (see Appendix A).

However, it turns out that such structures naturaly appear when studying compactifications of strata of the moduli space of Abelian differentials. Eskin, Kontsevich and Zorich show in a recent paper [4] that when a sequence of Abelian differentials (X_i, ω_i) converges to a boundary point in the Deligne-Munford compactification, then subsets $(Y_{i,j}, \omega_{i,j})$ corresponding to thick components of the X_i, after suitable rescaling converge to meromorphic differentials (see [4], Theorem 10). Smillie, in a work to appear, constructs a geometric compactification of the strata of the moduli space of Abelian differentials, by using only flat geometry, and where flat structures defined by meromorphic differentials are needed.

The connected components of the moduli space of Abelian differentials were described by Kontsevich and Zorich in [11]. They showed that each stratum has up to three connected component, which are described by two invariants: hyperellipticity and parity of spin structure, that arise under some conditions on the set of zeroes. Later, Lanneau has described the connected components of the moduli space of quadratic differentials. The main goal of the paper is to describe connected components of the moduli space of meromorphic differentials with prescribed poles and zeroes. It is well known that each stratum of the moduli space of genus zero meromorphic differentials is connected. We show that when the genus is greater than, or equal to two, there is an analogous classification as the one of Kontsevich and Zorich, while in genus one, there can be an arbitrarily large number of connected components.

In this paper, we will call *translation surface with poles* a translation surface that comes from a meromorphic differential on a puncture Riemann surface, where poles corresponds to the punctured points. We describe in Section 2 the local models for neighborhoods of poles. Similarly to the Abelian case, we denote by $\mathcal{H}(n_1,\dots,n_r,-p_1,\dots,-p_s)$ the moduli space of translation surfaces with poles that corresponds to meromorphic differentials with zeroes of degree $n_1,\dots,n_s$ and poles of degree $p_1,\dots,p_s$. It will be called *strata of the moduli space of meromorphic differentials*. We will always assume that $s>0$. A strata is nonempty as soon as $\sum_i n_i - \sum_j p_j = 2g-2$, for some nonnegative integer g and $\sum_j p_j > 1$.

For a genus one translation surface S with poles, we describe the connected components by using a geometric invariant easily computable in terms of the flat metric, that we call the *rotation number of a surface*. As we will see in Section 4, in the stratum $\mathcal{H}(n_1,\dots,n_r,-p_1,\dots,-p_s)$, the rotation number is a positive integer that divides all the n_i, p_j.

Theorem 1.1. *Let $\mathcal{H}(n_1,\dots,n_r,-p_1,\dots,-p_s)$, with $n_i,p_j>0$ and $\sum_j p_j > 1$ be a stratum of genus one meromorphic differentials. Denote by c be the number of positive divisors of $\gcd(n_1,\dots,n_r,p_1,\dots,p_s)$. The number of connected components of the stratum is:*

- *$c-1$ if $r=s=1$. In this case $n_1=p_1=\gcd(n_1,p_1)$ and each connected component corresponds to a a rotation number that divides n_1 and is not n_1.*
- *c otherwise. In this case each connected component corresponds to a rotation number that divides $\gcd(n_1,\dots,n_r,p_1,\dots,p_s)$.*

A consequence of the previous theorem, is that, contrary to the case of Abelian differentials, there can be an arbitrarily large number of

connected components for a stratum of meromorphic differentials (in genus 1). For instance, the stratum $\mathcal{H}(24,-24)$ has 7 connected components since the nontrivial divisors of 24 are $1, 2, 3, 4, 6, 8$ and 12.

The general classification uses analogous criteria as for Abelian differentials. We recall that in this case, the connected components are distinguished by the following (up to a few exception in low genera):

- *hyperellipticity*: if there is only one singularity or two singularities of equal degree, there is a component that consists only of hyperelliptic Riemann surfaces. For each translation surface, the hyperelliptic involution is an isometry. Slightly abusing with terminology, we usually call this component the *hyperelliptic component*.
- *parity of spin structure*: If all singularities are of even degree, there are two connected component (none of which is the hyperelliptic component) distinguished by a topological invariant easily computable in terms of the flat metric.

In Section 5, we define in our context the notion of hyperelliptic component and spin structure.

In the next theorem, we say that the set of poles and zeroes is:

- of *hyperelliptic type* if the degree of zeroes are or the kind $\{2n\}$ or $\{n, n\}$, for some positive integer n, and if the degree of the poles are of the kind $\{-2p\}$ or $\{-p,-p\}$, for some positive integer p.
- of *even type* if the degrees of zeroes are all even, and if the degrees of the poles are either all even, or are $\{-1,-1\}$.

Theorem 1.2. *Let $\mathcal{H} = \mathcal{H}(n_1, \dots, n_r, -p_1, \dots, -p_s)$, with $n_i, p_j > 0$ be a stratum of genus $g \geq 2$ meromorphic differentials. We have the following.*

1. *If $\sum_i p_i$ is odd and greater than two, then $\mathcal{H}$ is nonempty and connected.*
2. *If $\sum_i p_i = 2$ and $g = 2$, then:*
   - *if the set of poles and zeroes is of hyperelliptic type, then there are two connected components, one hyperelliptic, the other not (in this case, these two components are also distinghished by the parity of spin structure)*
   - *otherwise, the stratum is connected.*
3. *If $\sum_i p_i > 2$ or if $g > 2$, then:*
   - *if the set of poles and zeroes is of hyperelliptic type, there is exactly one hyperelliptic connected component, and one or two nonhyperelliptic components that are discribed below. Otherwise, there is no hyperelliptic component.*

- *if the set of poles and zeroes is of even type, then $\mathcal{H}$ contains exactly two nonhyperelliptic connected components that are distinguished by the parity of spin structure. Otherwise $\mathcal{H}$ contains exactly one nonhyperelliptic component.*

From the previous theorem, we see that there are at most three connected component in genus greater than or equal to two. For instance, the stratum $\mathcal{H}(4,4,-1,-1)$ contains a hyperelliptic connected component (zeroes and poles are of hyperelliptic type) and two nonhyperelliptic components (the zeroes are even and the poles are $\{-1,-1\}$). So it has three components. The stratum $\mathcal{H}(2,4,-1,-1,-2)$ is connected, since it does not have a hyperelliptic connected component and the poles and zeroes are not of even type.

The structure of the paper is the following:

- In Section 2, we describe general facts about the metric defined by a meromorphic differential and define a topology on the moduli space.
- In Section 3, we present three tools that are needed in the proof. The first two ones appear already in the paper of Kontsevich and Zorich, and in the paper of Lanneau. The third one is a version of the well known Veech construction for the case of translation surfaces with poles.
- In Section 4, we describe the connected components in the genus one case. Some of the results in genus one will be very usefull for the general genus.
- In Section 5, we describe the topological invariants for the general genus case, *i.e.* hyperelliptic connected components and the parity of spin structure.
- In Section 6, we compute the connected components for the minimal strata.
- In Section 7, we compute the connected components for the general case.

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2. Flat structures defined by a meromorphic differentials.

2.1. **Holomorphic one forms and flat structures.** Let X be a Riemann surface and let ω be a holomorphic one form. For each $z_0 \in X$ such that $\omega(z_0) \neq 0$, integrating ω in a neighborhood of z_0 gives local coordinates whose corresponding transition functions are translations, and therefore X inherits a flat metric, on $X \backslash \Sigma$, where Σ is the set of zeroes of ω.

In a neighborhood of an element of Σ, such metric admits a conical singularity of angle $(k+1)2\pi$, where k is the degree of the corresponding zero of ω. Indeed, a zero of degree k is given locally, in suitable coordinates by $\omega = (k+1)z^k dz$. This form is precisely the preimage of the constant form dz by the ramified covering $z \to z^{k+1}$. In terms of flat metric, it means that the flat metric defined locally by a zero of order k appear as a connected covering of order $k+1$ over a flat disk, ramified at zero.

When X is compact, the pair (X, ω), seen as a smooth surface with such translation atlas and conical singularities, is usually called a *translation surface*.

If ω is a meromorphic differential on a compact Riemann $\overline{X}$, we can consider the translation atlas defined defined by ω on $X = \overline{X} \backslash \Sigma'$, where Σ' is the set of poles of ω. We obtain a translation surface with infinite area. We will call such surface *translation surface with poles*.

Convention 2.1. *When speaking of a translation surface with poles $S = (X, \omega)$. The surface S equiped with the flat metric is noncompact. The underlying Riemann surface X is a punctured surface and ω is a holomorphic one-form on X. The corresponding closed Riemann surface is denoted as $\overline{X}$, and ω extends to a meromorphic differential on $\overline{X}$ whose set of poles are precisely $\overline{X} \backslash X$.*

Similarly to the case of Abelian differentials. A *saddle connection* is a geodesic segment that joins two conical singularities (or a conical singularity to itself) with no conical singularities on its interior.

We also recall that Riemann-Roch formula implies that $\sum_{i=1}^{r} n_i - \sum_{j=1}^{s} p_j = 2g-2$, where $\{n_1, \dots, n_s\}$ is the set (with multiplicities) of degree of zeroes of ω and $\{p_1, \dots, p_s\}$ is the set (with multiplicities) of degree of the poles of ω.

2.2. **Local model for poles.** The neighborhood of a pole of order one is an infinite cylinder with one end. Indeed, up to rescalling, the poles is given in local coordinates by $\omega = \frac{1}{z} dz$. Writing $z = e^{z'}$, we have $\omega = dz'$, and z' is in a infinite cylinder.

Now we describe the flat metric in a neighborhood of a pole of order $k \geq 2$. First, consider the meromorphic 1-form on $\mathbb{C} \cup \{\infty\}$ defined on $\mathbb{C}$ by $\omega = z^k dz$. Changing coordinates $w = 1/z$, we see that this form has a pole P of order $k+2$ at ∞, with zero residue. In terms of translation structure, a neighborhood of the pole is obtained by taking an infinite cone of angle $(k+1)2\pi$ and removing a compact neighborhood of the conical singularity. Since the residue is the only local invariant for a pole of order k, this gives a local model for a pole with zeroe residue.

Now, define $U_R = \{z \in \mathbb{C} ||z| > R\}$ equiped with the standard flat metric. Let V_R be the Riemann surface obtained after removing from U_R the π–neighborhood of the real half line $\mathbb{R}^-$, and identifying by the translation $z \to z + \imath 2\pi$ the lines $-\imath\pi + \mathbb{R}^-$ and $\imath\pi + \mathbb{R}^-$. The surface V_R is naturally equiped by a holomorphic one form ω coming from dz on V_R. We claim that this one form has a pole of order 2 at infinity and residue -1. Indeed, start from the one form on $U_{R'}$ defined by $(1 + 1/z)dz$ and integrate it. Choosing the usual determination of $\ln(z)$ on $\mathbb{C} \backslash \mathbb{R}^-$, one gets the map $z \to z + \ln(z)$ from $U_{R'} \backslash \mathbb{R}^-$ to $\mathbb{C}$, which extends to a injective holomorphic map f from $U_{R'}$ to V_R, if R' is large enough. Furthermore, the pullback of the form ω on V_R gives $(1 + 1/z)dz$. Then, the claim follows easily after the change of coordinate $w = 1/z$

Let $k \geq 2$. The pullback of the form $(1+1/z)dz$ by the map $z \to z^{k-1}$ gives $((k-1)z^{k-2} + (k-1)/z)dz$, *i.e.* we get at infinity a pole of order k with residue $-(k-1)$. In terms of flat metric, a neighborhood of a pole of order k and residue $-(k-1)$ is just the natural cyclic $(k-1)$–covering of V_R. A neighborhood of a pole of order k and a nonzero residue is done after proper rotating and rescaling.

For flat geometry, it will be convenient to forget the term $2\imath\pi$ when speaking of residue, hence we define the *flat residue* of a pole P to be $\int_{\gamma_P} \omega$, where γ_P is a small closed path that turns around a pole counterclokwise.

2.3. **Moduli space.** If (X, ω) and (X', ω') are such that there is a biholomorphism $f : X \to X'$ with $f^*\omega' = \omega$, then f is an isometry for the metrics defined by ω and ω'. Even more, for the local coordinates defined by ω, ω', the map f is in fact a translation.

As in the case of Abelian differentials, we consider the moduli space of meromorphic differentials, where (X, ω) (X', ω') if there is a biholomorphism $f : X \to X'$ such that $f^*\omega' = \omega$. A stratum corresponds to prescribed degree of zeroes and poles. We denote by $\mathcal{H}(n_1, \ldots, n_r, -p_1, \ldots, -p_s)$ the stratum that corresponds to meromorphic differentials with zeroes of degree $n_1, \ldots, n_r$ and poles of degree

$p_1, \ldots, p_s$. Such stratum is nonempty if and only if $\sum_{i=1}^{r} n_i - \sum_{j=1}^{s} p_j = 2g-2$ for some integer $g \geq 0$ and if $\sum_{j=1}^{s} p_j > 1$ (*i.e.* if there is not just one simple pole.).

We define the topology on this space in the following way: a small neighborhood of S, with conical singularities Σ, is defined to be the equivalent classes of surfaces S' for which there is a continuous injective map $f : S \backslash V(\Sigma) \to S'$ such that $V(\Sigma)$ is a (small) neighborhood of Σ, f is close to a translation, and the complement of the image of f is a union on disks. One can easily check that this topology is Hausdorff.

3. Tools

3.1. **Breaking up a singularity: local construction.** Here we describe a surgery, introduced by Eskin, Masur and Zorich (see [5], Section 8.1) for Abelian differentials, that "break up" a singularity of degree $k_1 + k_2 \geq 2$ into two singularities of degree $k_1 \geq 1$ and $k_2 \geq 1$ respectively. This surgery is local, since the metric is modified only in a neighborhood of the singularity of degree $k_1 + k_2$. In particular, it is also valid for the flat structure defined by a meromorphic differential.

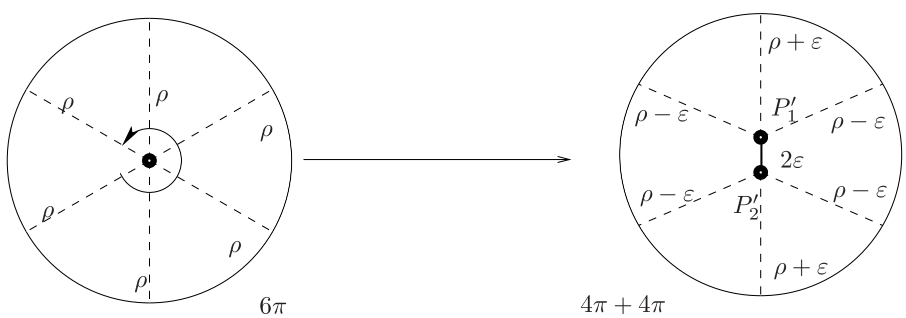

FIGURE 1. Breaking up a zero, after Eskin, Masur and Zorich

We start from a singularity of degree $k_1 + k_2$. A neighborhood of such singularity is obtained by gluing $(2k_1 + 2k_2 + 2)$ Euclidean half disks in a cyclic order. The singularity breaking procedure consists in changing continuously the way these half disks are glued together, as in Figure 1. This breaks the singularity of degree $k_1 + k_2$ into singularities of degree k_1 and k_2 respectively, and with a small saddle connection joining them.

3.2. **Bubbling a handle.** The following surgery was introduced by Kontsevich and Zorich in [11]. Since it is a local construction, it is also valid for meromorphic differentials. As before, we start from a

singularity of degree k_1+k_2 on a surface S. We first apply the previous surgery to get a pair of singularities of degree k_1 and k_2 respectively, and with a small saddle connection γ joining them. Then, we cut the surface along γ and obtain a flat surface with a boundary component that consists of two geodesic segments γ_1, γ_2. We identify their endpoints and the corresponding segments are now closed boundary geodescis γ_1', γ_2'. Then, we consider a cylinder with two boundary components isometric to γ_i', and glue each of these component to γ_i'. The angle between γ_1' and γ_2' is $(k_1+1)2\pi$ (and $(k_2+1)2\pi$)

Using a notation similar to the one introduced by Lanneau in [12], we will denote by $S\oplus(k_1+1)$ the resulting surface for an arbitrary choice of continuous parameters[1]. Different choices of continuous parameters leads to the same connected component and from a path $(S_t)_{t\in[0,1]}$, one can easily deduce a continuous path $S_t\oplus(k_1+1)$. Hence, as in [12], the connected component of $S\oplus s$ only depends on s and on the connected component of S. So, if S is in a connected component $\mathcal{C}$ of a stratum of Abelian (resp. meromorphic) differential with only one singularity, $\mathcal{C}\oplus s$ is the connected component of a stratum of Abelian (resp. meromorphic) differentials obtained by the construction.

The following lemma is Proposition 2.9 in the paper of Lanneau [12], written in our context. The ideas behind this proposition were also in the paper of Kontsevich and Zorich [11].

Lemma 3.1. *Let $\mathcal{C}$ be a minimal stratum of the moduli space of meromorphic differentials, with n the degree of the unique corresponding conical singularity. Then, the following statement holds.*

(1) $\mathcal{C}\oplus s_1\oplus s_2=\mathcal{C}\oplus s_2\oplus s_1$ *if* $1\leq s_1,s_2\leq n+1$ *and either* $s_1\neq\frac{n}{2}+1$ *or* $s_2\neq\frac{n+2}{2}+1$.
(2) $\mathcal{C}\oplus s_1\oplus s_2=\mathcal{C}\oplus s_2-1\oplus s_1+1$ *if* $1\leq s_1\leq n+1$ *and* $2\leq s_2\leq n+3$.
(3) $\mathcal{C}\oplus s_1\oplus s_2=\mathcal{C}\oplus s_2-2\oplus s_1$ *if* $1\leq s_1\leq n+1$ *and* $1\leq s_2\leq n+3$ *and* $s_2-s_1\geq 2$.
(4) $\mathcal{C}\oplus s=\mathcal{C}\oplus(n+2-s)$ *for all* $s\in\{1,\dots,n+1\}$

Remark 3.2. There is a small mistake in the statement of Lanneau: the condition “either $s_1\neq\frac{n}{2}+1$ or $s_2\neq\frac{n+2}{2}+1$” does not appear while it is necessary.

This leads to a gap in the proof of Lanneau's Lemma 6.13, but this problem is easily solved by using Lanneau's Lemma A.2.

[1]The notation slightly differs to the one introduced by Lanneau: since he manipulates *quadratic differentials*, the angles can be any multiples of π, while in our case, we only have multiples of 2π. So the surface we obtain would have been writen $S\oplus 2(k_1+1)$ with the notation of Lanneau.

3.3. **The infinite zippered rectangle construction.** In this section, we describe a construction of translation surfaces with poles which is analogous to the well known Veech zippered rectangle construction. We will call this construction the *infinite zippered rectangles construction.*

We first recall Veech construction.

3.3.1. *Veech construction of a translation surface.* The Veech construction, or zippered rectangle construction is usually seen as a way to define a suspension over an interval exchange map (see [17]). We can also see it as a easy way to define (almost any) translation surface. Consider a finite alphabet $\mathcal{A} = \{\alpha_1, \dots, \alpha_d\}$, and a pair on one to one maps $\pi_t, \pi_b : \mathcal{A} \to \{1, \dots, d\}$. Let $\zeta \in \mathbb{C}^{\mathcal{A}}$ be a vector for which each entry has positive real part.

The Veech construction can be seen in two (almost) equivalent ways. One with a $2d$ sided polygon, and one with d rectangles that are identified on their boundary.

We present the first one, which is simpler but not as general as the second one. Consider the broken line L_t on $\mathbb{C} = \mathbb{R}^2$ defined by concatenation of the vectors $\zeta_{\pi_t^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin. Similarly, we consider the broken line L_b defined by concatenation of the vectors $\zeta_{\pi_b^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin.

We assume that ζ is such that the vertices of L_t are always above the real line, except possibly the foremost right (and of course the one at the origin), and that similarly, the vertices of L_b are below the real line. Such ζ is called *suspension datum* (see [13]), and exists under a combinatorial condition on (π_t, π_b) usually called "irreducibility".

If the lines L_t and L_b have no intersections other than the endpoints, we can construct a translation surface X by identifying each side ζ_j on L_t with the side ζ_j on L_b by a translation. The resulting surface is a translation surface endowed with the form $\omega = dz$.

Remark 3.3. The surface so constructed can also be seen as a union of rectangles whose dimensions are easily deduced from π_t, π_b and ζ, and that are "zippered" on their boundary. One can define S directely in this way: the construction works also if L_t, L_b have other intersection points. This is the *zippered rectangle construction*, due to Veech ([17]). This construction coincides with the first one in the previous case.

3.3.2. *Basic domains.* Now we generalize the previous construction to obtain a flat surface that corresponds to a compact Riemann surfaces with a meromorphic differential. Instead of having a polygon with

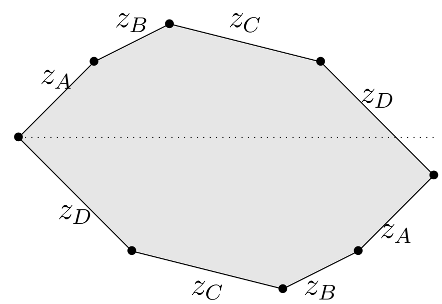

FIGURE 2. Veech construction of a translation surface

pairwise identification on its boundary, we will have a finite union of some "basic domains" which are half-planes and infinite cylinders with polygonal boundaries (see Figure 3).

Let $n \geq 0$. Let $\zeta \in \mathbb{C}^n$ be a complex vector whose entries have positive real part.

Consider the broken line L on $\mathbb{C}$ defined by concatenation of the following:

- the half-line l_1 that corresponds to $\mathbb{R}^-$,
- the broken line L_t defined as above, *i.e.* the concatenation of the segment defined by the vectors ζ_j (in this order) for $j = 1, \dots, n$ with starting point at the origin,
- the horizontal half line l_2 starting from the right end of L_t, and going to the right.

We consider the subset $D^+(z_1, \dots, z_n)$ (resp. $D^-(z_1, \dots, z_n)$) as the set of complex numbers that are above L. The line l_1 will be refered to as the *left* half-line, and l_2 will be refered to as the *right* half-line. We will sometime write such domains D^+ or D^- for short. The sets $D^\pm$ are kinds of half-planes with polygonal boundaries. Note that n might be equal to 0, and in this case, D^+ (resp. D^-) is just a half-plane with a marked point on its (horizontal) boundary.

Similarly, if $n \geq 1$, we can define the subset $C^+(z_1, \dots, z_n)$ (resp. $C^-(z_1, \dots, z_n)$) as the set of complex numbers that are above L_t. Its boundary consists of two infinite vertical half-lines joined by the broken line L_t. The two infinite half-lines will be identified in the resulting construction, hence $C^\pm$ is just an infinite half-cylinder with a polygonal boundary.

3.3.3. *An example: a surface with a single pole of order 2.* The idea of the construction is to glue by translation the basic domains together in order to get a noncompact translation surface with no boundary

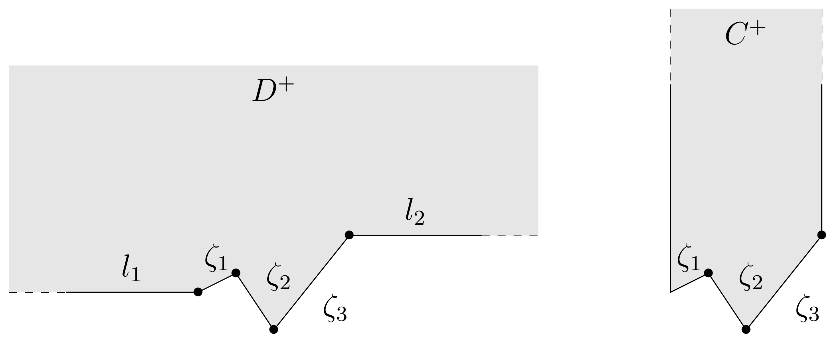

FIGURE 3. A domain $D^+(\zeta_1, \zeta_2, \zeta_3)$ and a domain $C^+(\zeta_1, \zeta_2, \zeta_3)$

components. Since the formal description is rather technical, we first present a simple version of the construction.

Let $\mathcal{A}$ be a finite alphabet and $\pi_t, \pi_b : \mathcal{A} \to \{1, \dots, d\}$ be one-to-one maps. Let $\zeta \in \mathbb{C}^{\mathcal{A}}$ be such that $Re(\zeta_\alpha) > 0$ for all $\alpha \in \mathcal{A}$.

We define a flat surface S as the disjoint union of the two half-planes $D^+ = D^+(\zeta_{\pi_t^{-1}(1)}, \dots, \zeta_{\pi_t^{-1}(n)})$ and $D^- = D^-(\zeta_{\pi_b^{-1}(1)}, \dots, \zeta_{\pi_b^{-1}(n)})$ glued on their boundary by translation: the left half line of D^+ being glued to the left half-line of D^- and similarly with the right half-lines, and a segment in D^+ corresponding to some ζ_i is glued to the corresponding one of D^-.

Note that contrary to the case of compact translation surfaces, there is no "suspension data condition" on ζ, hence, no combinatorial condition on π. The only condition that we require is that $Re(\zeta_i) > 0$ for all i. Note also that we can have $n = 0$, in this case $S = \mathbb{C}$.

3.3.4. *General case.* We can generalize the above construction in order to have several poles of any order. Instead of considering two half-planes $D^+, D-$, we will do the same construction starting from $2d$ half-planes, $s^+ + s^-$ infinite cylinders, and define identification on their boundary. More precisely:

Let $\zeta \in \mathbb{C}^n$ with positive real part. We consider the following combinatorial data:

- A collection $\mathbf{n}^+$ of integers $0 = n_0^+ \leq n_1^+ \leq \dots n_d^+ < \dots < n_{d+s^+}^+ = n$
- A collection $\mathbf{n}^-$ of integers $0 = n_0^- \leq n_1^- \leq \dots n_d^- < \dots < n_{d+s^-}^- = n$
- A pair of maps $\pi_t, \pi_b : \mathcal{A} \to \{1, \dots, n\}$
- A collection $\mathbf{d}$ of integers $0 = d_0 < d_1 < d_2 < \dots < d_r = d$.

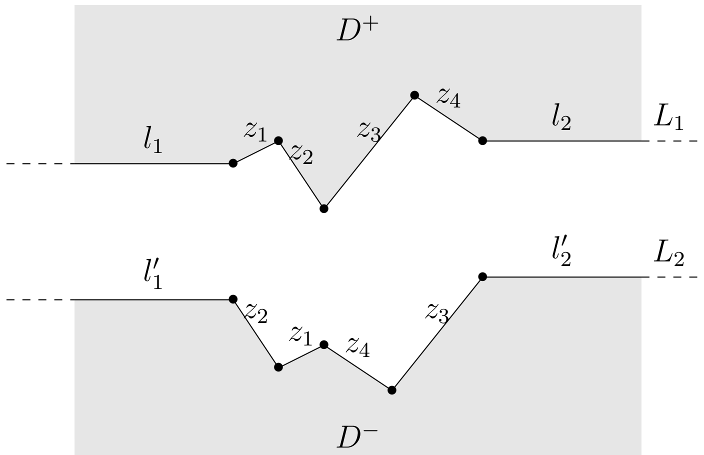

FIGURE 4. Construction of a translation surface with a degree 2 pole.

The resulting surface will have r poles of order greater than or equal to two, and $s^+ + s^-$ poles of order 1.

For each $i \in \{0, \ldots, d-1\}$, we consider the basic domains as defined before $D_i^+(\zeta_{\pi_t^{-1}(n_i^+ +1)}, \ldots, \zeta_{\pi_t^{-1}(n_{i+1}^+)})$ and $D_i^-(\zeta_{\pi_b^{-1}(n_i^- +1)}, \ldots, \zeta_{\pi_b^{-1}(n_{i+1}^-)})$. For $i \in \{d, \ldots, d+s^+ -1\}$, we define $C_i^+(\zeta_{\pi_t^{-1}(n_i^+ +1)}, \ldots, \zeta_{\pi_t^{-1}(n_{i+1}^+)})$. For $i \in \{d, \ldots, d+s^- -1\}$, we define $C_i^-(\zeta_{\pi_b^{-1}(n_i^- +1)}, \ldots, \zeta_{\pi_b^{-1}(n_{i+1}^-)})$.

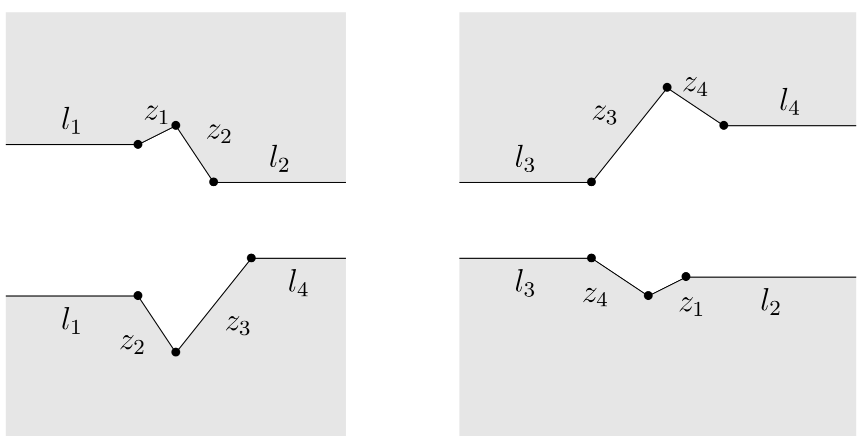

FIGURE 5. Construction of a translation surface with a degree 3 pole.

Then, we glue these domains together on their boundary by translations:

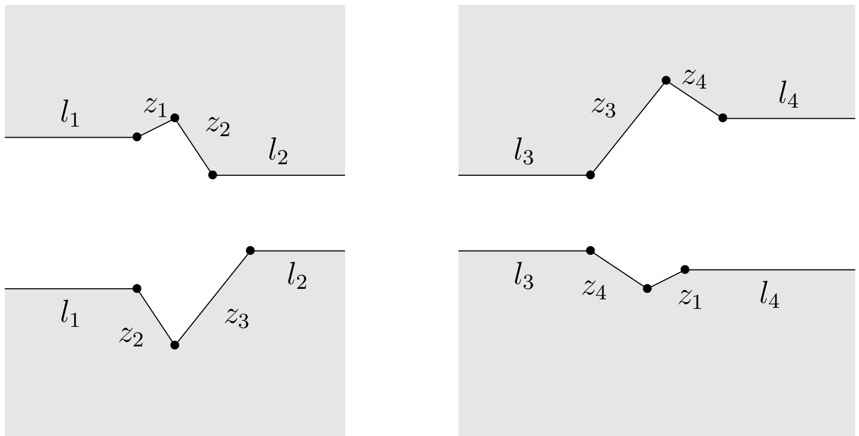

FIGURE 6. Construction of a translation surface with two poles of degree 2

- each segment corresponding to a parameter ζ_i in a "$+''$ domain is glued to the unique corresponding one in a "$-''$ domain.
- each left line of a domain D_i^+ is glued to the left line of the domain D_i^-.
- For each $i \in \{1, \dots, d\} \backslash \{d_1, \dots, d_r\}$ the right line of the domain D_i^- is glued to one of the domain D_{i+1}^+.
- For each $i = d_k$, $k > 0$, the right line of the domain D_i^- is glued to one of the domain $D_{d_{k-1}+1}^+$.
- For each C_i^+ and C_i^-, the two vertical lines are glued together.

The resulting surface S has no more boundary components, and is a flat surface with poles and conical singularities. It might not be connected for a given combinatorial data. We will consider only those that that give connected surfaces.

Note that such surface is easily decomposes into a finite union of vertical strips and half-planes with vertical boundary (*i.e.* "infinite rectangle"), that are "zippered" on their boundary.

Example 3.4. Figure 5 shows an example with $d = 2$, $s^+ = s^- = 0$, $n^+ = (0, 2, 4)$, $n^- = (0, 2, 4)$, $\pi_t = Id$, $(\pi_b^{-1}(1), \dots, \pi_b^{-1}(n)) = (2, 3, 4, 1)$ and $\mathbf{d} = (0, 2)$. One gets a surface in $\mathcal{H}(-3, 5)$.

Figure 6 shows an example with the same data, except that $\mathbf{d} = (0, 1, 2)$. One gets a surface in $\mathcal{H}(-2, -2, 2, 2)$.

Lemma 3.5. *Let S be a genus g surface in $\mathcal{H}(n_1, \dots, n_r, -p_1, \dots, -p_s)$, obtained by the infinite zippered rectangle construction with a continuous parameter $\zeta \in \mathbb{C}^n$. Then*

$$n = 2g + r + s - 2$$

Proof. By construction, the surface (pole included) is obtained by gluing s disks on their boundary. The resulting surfaces admits a decompositions into cells, with s faces, n edges, and r vertices. So, we have $2-2g = s-n+r$, and the result follows. □

The following proposition will be very usefull in the remaing of the paper. It is analogous to fact that any translation surface with no vertical saddle connection is obtained by the Veech construction.

Proposition 3.6. *Any translation surface with poles with no vertical saddle connection is obtained by the infinite zippered rectangle construction.*

Proof. According to the book of Strebel [16] Section 11.4, when there are no vertical saddle connection the vertical trajectories one of the following to kind:

(1) line that converges to a pole in each directions.
(2) half-lines starting (or ending) from a conical singularity and converging to a pole on their infinite direction.

Furthermore, the set of non-singular vertical trajectories is a disjoint union of half-planes and of vertical strips $\sqcup_i \mathcal{P}_i \sqcup_j \mathcal{S}_j$. The half-planes have one vertical boundary component, and the strips have two vertical boundary component. We choose these half-planes or strips as maximals, so each boundary component necessarily contains a conical singularity. This singularity is unique for each connected component, otherwise there would be a vertical saddle connection on the surface. This number of half-planes and strips is necessarily finite, since there is only a finite number of conical singularities, and each conical singularities is adjacent to a finite number of half-plane or strip.

We cut each half-plane $\mathcal{P}_i$ in the following way: the boundary of $\mathcal{P}_i$ consists of a union of two vertical half-lines starting from the conical singularity. We consider the unique horizontal half-line starting from this singularity and cut $\mathcal{P}_i$ along this half line. It decomposes $\mathcal{P}_i$ into two components $\mathcal{P}_i^+$ (the upper one) and $\mathcal{P}_i^-$ (the lower one).

We cut each strip $\mathcal{S}_j$ in the following way: the boundary of $\mathcal{S}_i$ consists has two components, each consists of a union of two vertical half-lines starting from a conical singularity. There is a unique geodesic segment joining these two boundary singularities. We cut $\mathcal{S}_j$ along this segment and obtain two components $\mathcal{S}_j^+$ and $\mathcal{S}_j^-$.

The surface S is obtained as the disjoint union of the $\mathcal{S}_j^\pm$ and $\mathcal{P}_i^\pm$, glued to each other by translation on their boundary components. Now we remark that the $\mathcal{P}_i^+$ and $\mathcal{S}_j^+$ are necessarily glued to each other along their vertical boundary components. Under this identification,

$\sqcup \mathcal{P}_i^+ \sqcup_j \mathcal{S}_j^+$ is a union of subsets of the type D^+ and S^+, as in the previous construction.

Similarly, gluing together along vertical sides the union of the $\mathcal{S}_j^-$ and $\mathcal{P}_i^-$, one obtain a union of D^- and C^- type subsets.

So the surface is obtained by the infinite zippered rectangle construction. □

Remark 3.7. Note that the parameters $(\zeta_i)_i$ are uniquely defined (once the suitable vertical direction is fixed) and the infinite zipperered rectangle construction defines a triangulation of the surface for which the (ζ_i) are the local parameters for the strata. Hence, map $S \to (\zeta_i)_i$ is a local homeomorphism. The corresponding saddle connections form a basis of the relative homology $H_1(S, \Sigma, \mathbb{Z})$, where Σ is the set of conical singularities of S.

Note that for any translation surface with poles, the set of saddle connection is at most countable, so it is always possible to rotate the surface in such way that there are no vertical saddle connection. Hence the previous theorem gives a representation for *any* translation surface with pole. One important consequence this theorem is Proposition 7.1, which is the analogous version of a key argument in [11].

4. Genus one case

4.1. **Connected components.** We first recall the following result in algebraic geometry, which is a consequence of the Abel theorem.

Theorem. *Let $\overline{X} = \mathbb{C}/\Gamma$ be a torus and let $D = \sum_i \alpha_i P_i$ is a divisor. Then there exists a meromorphic differential with D as a divisor if and only if $\sum z_i \in \Gamma$, where for each i, z_i is a representative in $\mathbb{C}$ of P_i.*

Now we use this theorem to describe the connected components in the genus one case.

Theorem 4.1. *Let $\mathcal{H} = \mathcal{H}(n_1, \dots, n_r, -p_1, \dots, -p_s)$ be a stratum of genus one meromorphic differentials. Let $d = \gcd(n_1, \dots, n_r, p_1, \dots p_s)$, and let c be the number of positive divisors of d. Then the number of connected components of $\mathcal{H}$ is:*

- *c if $r + s \geq 3$.*
- *$c - 1$ if $r + s = 2$.*

Proof. According to the previous theorem, an element in $\mathcal{H}$ is given, up a scalar multiple, by a pair $(\overline{X}, D)$, where $\overline{X}$ is in the moduli space of genus one Riemann surfaces $\mathcal{M}_1$, and D is a divisor on $\overline{X}$. So, $\mathcal{H}$ (or its projectivization) is a covering of $\mathcal{M}_1$.

We first assume that $r+s=2$. Then $\mathcal{H}=\mathcal{H}(n,-n)$, for some $n\geq 2$ (the stratum $\mathcal{H}(1,-1)$ is empty). We have $d=n$. Fix $\overline{X}_0\in\mathcal{M}_1$ a regular point, and choose v_1,v_2 such that $\overline{X}_0=\mathbb{C}/(v_1\mathbb{Z}+v_2\mathbb{Z})$. For each $(\overline{X}_0,\omega)\in\mathcal{H}$, the differential ω is uniquely defined, up to a multiple constant, by its divisor. One can assume that the divisor is $-n.0+nQ$, where Q is represented in a unique way by a complex number of the form $\frac{p}{n}v_1+\frac{q}{n}v_2$, with $(p,q)\neq(0,0)$ and $0\leq p,q<n$. Since $\overline{X}_0$ is taken regular, there is a one to one correspondance with elements of $\mathcal{H}$ of underlying surface $\overline{X}_0$, and such pairs (p,q) of integers.

The monodromy of the covering $P\mathcal{H}\to\mathcal{M}_1$ is given by the two maps $\phi_1:(p,q)\to(p+q,q) \mod n$ and $\phi_2:(p,q)\to(p,q+p) \mod n$. We remark that $d'=\gcd(p,q,n)$ is invariant by this action and the condition on (p,q) implies that $0<d'<n$. Hence is an invariant of the connected components of $\mathcal{H}$. The number of possible d' is $c-1$. We claim that each (p,q) as a (unique) representative modulo these actions of the kind $(d',0)$. To prove the claim, we start from an element (p,q) and do an algorithm similar to Euclid's algorithm. Without loss of generality, one can assume that $p\neq 0$ and $q\neq 0$. Applying ϕ_1^r for some well chosen r, we can obtain (p',q) with $0<p'\leq\gcd(q,n)$. Similarly, we can obtain (p,q') with $0<q<\gcd(p,n)$. So if either $\gcd(q,n)<p$ or $\gcd(p,n)<q$, we obtain (p',q') with $p'+q'<p+q$. Otherwise $p\leq\gcd(q,n)\leq q$ and $q\leq\gcd(p,n)\leq p$. This implies $p=q=d'$, and the result follows.

Now we assume $r+q\geq 3$. We proceed in the same way as before: we fix $\overline{X}_0=\mathbb{C}/\Gamma$ and a basis v_1,v_2 of Γ. Then a meromorphic differential is given by a a vector $(z_1,\dots z_r,z_1',\dots,z_s')\mathbb{C}^{r+s}$ with pairwise disjoint entries (modulo Γ), and satisfying the linear equality $\sum_{i=1}^r n_iz_i-\sum_{i=1}^s p_iz_i=pv_1+qv_2$. One can remark that:

- For each (p,q) the set of $(z_i)_i,(z_j')_j$ satisfying the previous condition is nonempty and connected.
- If we choose other representatives z_i,z_j' for the same differential ω, this changes (p,q) by $(p+\sum_i\alpha_in_i+\sum_j\beta_jp_j,q+\sum_i\alpha_i'n_i+\sum_j\beta_j'p_j)$, where $(\alpha_i,\beta_j,\alpha_i',\beta_j')$ can be any integers.
- The action of the the two generators of the modular group changes (p,q) by $(p+q,q)$ and $(p,q+p)$ respectively.

Then, by a proof very similar to the previous one, one can see that $d'=\gcd(p,q,n_1,\dots,n_r,p_1,\dots,p_s)$ is an invariant of connected component and one can find representative in each connected component satisfying $(p,q)=(d',0)$. So the number of connected component is precisely c.

Note that the difference with the first case is that any pair $(p,q)\in\mathbb{Z}^2$ is possible.

□

4.2. **Flat point of view: rotation number.** The previous section classifies the connected component of the moduli space of meromorphic differentials in the genus one case from an complex analytic point of view.

But the invariant which is given is not easy to describe in terms of flat geometry. The next theorem gives an interpretation in terms of flat geometry.

Let γ be a simple closed curve parametrized by the arc length on a translation surface that avoids the singularities. Then $t \to \gamma'(t)$ defines a map from $\mathbb{S}^1$ to $\mathbb{S}^1$. We denote by $Ind(\gamma)$ the index of this map.

Definition 4.2. Let $S = (X,\omega) \in \mathcal{H}(n_1,\dots,n_r,-p_1,\dots-p_s)$ be a genus one translation surface with poles. Let (a,b) be a symplectic basis of the homology the underlying compact Riemann surface $\overline{X}$ and γ_a,γ_b be arc-length representatives of a,b, with no self intersections, and that avoid the zeroes and poles of ω. We define the *rotation number* of S to by:

$$rot(S) = \gcd(Ind(\gamma_a), Ind(\gamma_b), n_1,\dots n_r, p_1,\dots,p_s)$$

Theorem 4.3. *Let $\mathcal{H} = \mathcal{H}(n_1,\dots,n_r,-p_1,\dots-p_s)$ be a stratum of genus 1 meromorphic differentials. The rotation number is an invariant of connected components of $\mathcal{H}$.*

Any positive integer d which divide $\gcd(n_1,\dots,n_r,p_1,\dots p_s)$ is realised by a unique connected component of $\mathcal{H}$, except for the case $\mathcal{H} = \mathcal{H}(n,-n)$ where $d=n$ doesn't occur.

Proof. Let (a,b) be a symplectic basis of $H_1(X,\mathbb{Z})$. Let γ_a,γ_a' be representatives of a that are simple closed curves and doesn't contain a singularity. Since $\overline{X}$ is a torus, γ_a and γ_a' are homotopic as curves defined on $\overline{X}$. The index of γ_a doesn't change while we deform γ_a without crossing a pole or a zero. It is easy to see that when crossing a singularity of order $k\in\mathbb{Z}$, the index is changed by adding $\pm k$. Hence the rotation number only depend on the homology class of a and b.

If γ_a and γ_b intersects in one point, then there is a standard way to construct a simple closed curve representing $a\pm b$. Its index is $Ind(\gamma_a)\pm Ind(\gamma_b)$, and we obtain representatives of the symplectic basis $(a\pm b,b)$ (or $(a,a\pm b)$). The rotation number doesn't change by this operation. With this procedure, we can obtain any other symplectic basis of $\overline{X}$.

Hence the rotation number is well defined for a given element of $\mathcal{H}$. Also, it is invariant by deforming $(\overline{X},\omega)$ inside the ambiant stratum,

since by continuous deformation, we can keep track of a pair of representatives of a basis, and the indices are constant under continuous deformations.

To prove the last part of the theorem, we remark that a surface in $\mathcal{H}(n, -p_1, \dots, -p_s)$ obtained from $H(n-2, -p_1, \dots, -p_s)$ by bubbling a handle with parameter $k \in \{1, \dots, n-1\}$ has a rotation number equal to $\gcd(k, p_1, \dots, p_s)$ by a direct computation. Since $k < n$, we have $\gcd(k, p_1, \dots, p_s) < n$ so n is never a rotation number. Now we blow up the singularity of order n to get r singularities of order $n_1, \dots, n_r$. Since this doesn't change the metric outside a small neighborhood of the singularity of order n, we obtain a rotation number equal to $\gcd(k, n_1, \dots, n_r, p_1, \dots, p_s)$.

The previous construction gives at least as many connected component as the number given by Theorem 4.1. So, we see each rotation number is realized by a unique component, and that this component is realized by the bubbling a handle construction. □

Note that the last two paragraphs of the proof of the last theorem gives the following description of the connected components of the minimal strata in genus one.

Proposition 4.4. *Let $\mathcal{H} = \mathcal{H}(n, -p_1, \dots, -p_s)$ be a minimal stratum of genus one meromorphic differentials. Any connected component of $\mathcal{H}$ is obtained after bubbling a handle.*

Also, for $1 \leq k_1, k_2 \leq n-1$ we have:

$$\mathcal{H}(n-2, -p_1, \dots, -p_s) \oplus k_1 = \mathcal{H}(n-2, -p_1, \dots, -p_s) \oplus k_2$$

if and only if $\gcd(k_1, p_1, \dots, p_s) = \gcd(k_2, p_1, \dots, p_s)$.

Remark 4.5. It is shown in the appendix that there are some translation surface with pole that do not contain any closed geodesic.

5. Spin structure and hyperelliptic components

Recall that in the classification of the connected component of strata of the moduli space of Abelian differentials [11], the connected components are distinghished by two invariant.

- "Hyperelliptic components": there are some connected components whose corresponding translation surface have all an extra symmetry.
- "Parity of spin structure", which is a complex invariant that can be expressed in terms of the flat geometry by a simple formula.

5.1. **Hyperelliptic components.**

Definition 5.1. A translation surface with poles S is said to be *hyperelliptic* if there exists an isometric involution $\tau : S \to S$ such that S/τ is a sphere. Equivalently, the underlying Riemann surface $\overline{X}$ is hyperelliptic and the hyperelliptic involution τ satisfies $\tau^*\omega = -\omega$.

Remark 5.2. In the case of Abelian differentials, if the underlying Riemann surface is hyperelliptic, then the translation surface is hyperelliptic since there are no nonzero holomorphic one forms on the sphere. In our case, similarly to the case of quadratic differentials, the underlying Riemann surface might be hyperelliptic, while the corresponding translation surface is not.

Proposition 5.3. *Let n, p be positive integers with $n \geq p$. The following strata admit a connected component that consists only of hyperelliptic translation surfaces.*

- $\mathcal{H}(2n, -2p)$
- $\mathcal{H}(2n, -p, -p)$
- $\mathcal{H}(n, n, -2p)$
- $\mathcal{H}(n, n, -p, -p)$

Furthermore, any strata that contains an open set of flat surfaces with a nontrivial isometric involution is in the previous list for some $n \geq p \geq 1$.

Proof. Let $\mathcal{H}$ be a stratum and $\mathring{\mathcal{H}}^{hyp} \subset \mathcal{H}$ the interior of the set of elements of $\mathcal{H}$ that admit an nontrivial isometric involution.

Given a combinatoria data $\sigma = (\mathbf{n}^+, \mathbf{n}^-, \pi_t, \pi_b, \mathbf{d})$ that defines an infinite zippered rectangle construction, we denote by $\mathcal{C}_\sigma$ the set of flat surfaces that are obtained by this construction with parameter σ, up to a rotation. Clearly, $\mathcal{C}_\sigma$ is open and connected.

We claim that for each σ, the intersection between $\mathcal{C}_\sigma$ and $\mathring{\mathcal{H}}^{hyp}$ is either $\mathcal{C}_\sigma$ or empty. Indeed, choose a generic parameter ζ for the infinite zippered rectangle construction, such that the corresponding surface $S(\sigma, \zeta)$ is in $\mathring{\mathcal{H}}^{hyp}$. Let $D^+(z_1, \dots, z_k) \subset S(\sigma, \zeta)$ be a half-plane of the construction. Then, ζ being generic, an isometric involution τ will necessarily send the segment corresponding to z_i to itself. Hence if τ is not the identity, it is easy to see that the set $D^+(z_1, \dots, z_k)$ will be sent to $D^-(z_k, z_{k-1}, \dots, z_1)$, and therefore, we can define a similar involution for *any* value of $z_1, \dots, z_k$. Since this argument is valid for any $D^{\sum_{pm}}$ and $C^\pm$ components, we see that all flat surfaces obtained by the infinite zippered rectangle construction with combinatorial datum σ have a nontrivial isometric involution. This proves the claim.

Now we remark that, by Proposition 3.6, $\mathcal{H} = \cup_\sigma \mathcal{C}_\sigma$, where the union is taken on all σ that corresponds to $\mathcal{H}$. The previous claim implies that $\mathring{\mathcal{H}}^{hyp}$ and its complement in $\mathcal{H}$ are both unions of some $\mathcal{C}_\sigma$, so if $\mathring{\mathcal{H}}^{hyp}$ is nonempty, it is a connected component of $\mathcal{H}$.

Now we check that if $\mathring{\mathcal{H}}^{hyp}$ is not empty, then the stratum $\mathcal{H}$ is in the given list, *i.e.* there is either one even degree zero (resp. pole) or two equal degree zeroes (res. poles). Let $\zeta_1, \dots, \zeta_n$ be the continuous data in the infinite zippered rectangle construction for an element S in $\mathring{\mathcal{H}}^{hyp}$. The above condition implies that for each ζ_i, the middle of the corresponding segment in the surface is a fixed point for the involution τ. So, there are at least n fixed points. Let r be the number of conical singularities , s be the nomber of poles and let g' be the genus of S/τ. We must have $\#(Fix(\tau)) = 2g + 2 - 4g'$, and $2g + r + s - 2 = n \leq \#(Fix(\tau))$ (see Lemma 3.5). Since $r, s > 1$, this implies $g' = 0$, so S is hyperelliptic, and $\#(Fix(\tau)) - n = 4 - r - s$. The fixed points of τ in $\overline{X}$ that do not correspond to the middle of a z_i segment are necessarily either conical singularities or pole.

The above combinatorial condition on the infinite zippered rectangle construction implies that S has either two equal degree poles that are interchanged by τ or one pole of even degree that is preserved by τ. So the condition $\#(Fix(\tau)) - n = 4 - r - s$ implies that either there is one conical singularity which is fixed by τ, or there are two singularities P_1, P_2 that are not fixed by τ. By a similar argument as in the proof of Proposition 7.1, P_1, P_2 are the endpoints of a saddle connection corresponding to a parameter ζ_i, so they are interchanged by τ, hence they are of the same degree. Therefore, the stratum is necessarily one of the given list.

The last step of the proof is to check that for the strata given in the statement, $\mathring{\mathcal{H}}^{hyp}$ is nonempty. This is an elementary check by using the infinite zippered rectangle construction that satisfies the previous condition.

□

5.2. **Parity of spin structure.** Let $S = (X, \omega)$ be a (standard) translation surface and $(\alpha_i, \beta_i)_i$ be a symplectic basis for the homology. According to Kontsevich and Zorich [11], the parity of spin structure for S is given by the formula:

$$\sum_i (ind(\alpha_i) + 1)(ind(\beta_i) + 1) \mod (2)$$

Therefore, the number defined by above formula doesn't depend on the choice of the basis. It is clear that it doesn't change under small

deformation of the surface inside the ambiant stratum, hence is an invariant of connected components of strata.

We will define a similar formula in our case and show that it doesn't depend on the choice of the basis, and still call it *parity of spin structure*. The key argument is to reduce the problem to the case of standard translation surfaces. We define the following:

Definition 5.4. Let S be a translation surface with poles, and $\widetilde{S}$ be a standard translation surface. We say that $\widetilde{S}$ is a *closing* of S if S isometrically embeds in $\widetilde{S}$, once removed a neighborhood of its poles, and such that the complement of the image of the map in $\widetilde{S}$ is connected.

A closing is not necessarily unique, but, as shown in the next lemma, it always exists.

Lemma 5.5. *Any translation surface with poles admits a closing.*

Proof. Recall that we call *flat residue* of a pole P the value $2\imath\pi Res(P)$.

Let $S \in \mathcal{H}(n_1, \dots, n_r, -p_1, \dots, -p_s)$. We assume for simplicity that all residues are non zero. The case with zero residue is similar. Up to rotating S, we can assume that all residues have nonzero real parts. Let $P_1, \dots, P_s$ be the poles of S of order $p_1, \dots, p_s$ respectively.

For each $i \in \{1, \dots, s\}$ we define $p'_i = p_i - 2$ if $p_i \geq 2$ and $p'_i = 0$ if $p_i = 1$. Now we use the (usual) Veech construction defined in Section 3.3.1 to construct a flat surface in $\mathcal{H}(p'_1, \dots, p'_s, k)$, where k is any positive integer such that $\mathcal{H}(p'_1, \dots, p'_s, k)$ is nonempty.

Let $(\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$ and a permutation $\pi \in \Sigma_n$ as in Section 3.3.1. The Veech construction defines a surface S_0. We can assume that all $|\zeta_i|$ are very large. Recall that S_0 is obtained in the following way: we consider the broken line L_t defined by concatenation of the vectors ζ_j (for $j \in \{1, \dots, n\}$), and the broken L_b starting from the same point as L_t, and obtained by the concatenation of the vectors $\zeta_{\pi^{-1}(j)}$. We denote by $Q_1, \dots, Q_s$ the (pairwise distinct) singularities of degree $p'_1, \dots, p'_s$ respectively. Now for each $i \in \{1, \dots, s\}$, if P_i has flat residue η_i with negative real part, we choose $j \in \{1, \dots, n\}$ such that the vertex of L_t corresponding to the right endpoint of ζ_j corresponds to P_i. Then we change L_t by pasting in a segment of vector $-\eta_i$ between ζ_j and ζ_{j+1}. If P_i has a flat residue η_i with positive real part, we choose $j \in \{1, \dots, n\}$ such that the vertex of L_b corresponding to the right endpoint of $\zeta_{\pi^{-1}(j)}$ corresponds to P_i. Then we change L_b by pasting in a segment of vector η_i between $\zeta_{\pi^{-1}(j)}$ and $\zeta_{\pi^{-1}(j+1)}$. The two newly created broken lines L'_t, L'_b still share the same starting point and ending point since $\sum_i \eta_i = 0$. Also, if the ζ_j are large enough compared to the η_i, the lines L'_t, L'_b still only intersect at their starting and ending

points, so they form a polygon $\mathcal{P}'$ with $2n+s$ sides, and where $2n$ of these sides are pairwise parallel and of the same length. Identifying the sides by translation, we obtain a surface with boundary S'. Each boundary component is isometric to a segment corresponding to a η_i or a $-\eta_i$, and corresponds to the pole P_i.

Let $i \in \{1, \dots, s\}$. We assume that P_i is a pole of order greater than, or equal to two. Let s_i be the boundary segment of S' in the previous construction. Let V_δ be the δ–neighborhood of s_i in S'. Note that, by construction, $\int_{\partial V_\delta} \omega = -\eta_i$. If δ is large enough, ∂V_δ is isometric to a curve γ_i in S that turns around the pole P_i in the clockwise direction. Hence we can glue $S' \backslash V_\delta$ with the interior of γ_i in S.

If P_i is a pole or order one, its boundary is an infinite cylinder. We cut this cylinder along a closed geodesic and remove the pole. The resulting boundary component is isometric to s_i and can be glued to the corresponding boundary component of S'.

One gets a closing of S by doing this construction for all values of i. $\square$

Let $S = (X, \omega)$ be a translation surface with poles and let $(\alpha_i, \beta_i)_i$ be a collection of curves representing a symplectic basis of $H_1(\overline{X}, \mathbb{Z})$, and that avoid the zeroes and poles of ω. By analogy to the formula of Kontsevich and Zorich, we define *the parity of spin structure* of S by:

$$\sum_i (ind(\alpha_i)+1)(ind(\beta_i)+1) \mod (2)$$

Proposition 5.6. *Let S be a translation surface with poles such that all singularities are of even degree. The parity of spin structure of S as defined above doesn't depend on the choice of the symplectic basis.*

Proof. By the construction of the previous lemma, there is a closing of S that contains only even singularity. Let $\mathring{S}$ be a subset of S obtained by removing a neighborhood of the poles, and such that $\mathring{S}$ embeds in $\widetilde{S}$. Without loss of generality, we can assume that the paths $(\alpha_i, \beta_i)_{i \in \{1, \dots, g\}}$ on S defining the symplectic basis are paths on $\mathring{S}$, and therefore define a symplectic family of $H_1(\widetilde{S}, \mathbb{Z})$.

Now we complete $(\alpha_i, \beta_i)_{i \in \{1, \dots, g\}}$ into a symplectic basis of $H_1(\widetilde{S}, \mathbb{Z})$ in the following way: let g' be the genus of $\widetilde{S} \backslash \mathring{S}$ and choose representatives $(\alpha_j, \beta_j)_{j \in \{g+1, \dots, g+g'\}}$ for a maximal symplectic family of $\widetilde{S} \backslash \mathring{S}$. The family $(\alpha_i, \beta_i)_{i \in \{1, \dots, g+g'\}}$ is symplectic but not a basis. The surface $\mathring{S} \subset \widetilde{S}$ has r connected components. Choose $r-1$ of such components and consider the corresponding curves $\alpha_{g+g'+1}, \dots, \alpha_{g+g'+r-1}$. Now it is

easy to complete to get a symplectic basis of $H_1(\widetilde{S},\mathbb{Z})$ them by choosing suitable dual paths β_j for the α_j.

Now let $(\alpha_i',\beta_i')_{i\in\{1,\dots,g\}}$ be another basis of S. We do the same construction as before an get another symplectic basis $(\alpha_i',\beta_i')_{i\in\{1,\dots,g+g'+r-1\}}$. We can choose $\alpha_i=\alpha_i'$ for $i\in\{g+1,\dots,g+g'+r-1\}$ and $\beta_i=\beta_i'$ for $i\in\{g+1,\dots,g+g'\}$ but not for $i\in\{g+g'+1,\dots,g+g'+r-1\}$. Since the parity of the spin structure for the surface $\widetilde{S}$ does not depend on the choice of the basis by the result of [11], we have:

$$\begin{aligned}\sum_{i=1}^{g}&(ind(\alpha_i)+1)(ind(\beta_i)+1)-\sum_{i=1}^{g}(ind(\alpha_i')+1)(ind(\beta_i')+1)\\ &=-\sum_{i=g+1}^{g+g'+r-1}(ind(\alpha_i)+1)(ind(\beta_i)-ind(\beta_i'))\mod 2\\ &=-\sum_{i=g+g'+1}^{g+g'+r-1}(ind(\alpha_i)+1)(ind(\beta_i)-ind(\beta_i'))\mod 2\end{aligned}$$

Since the degree is even, it is easy to see that $ind(\alpha_i)$ is odd for $i\in\{g+g'+1,\dots,g+g'+r-1\}$, so the term on the right in the previous equation is zero. This proves the proposition. □

5.3. **Spin structure for surfaces with two simple poles.** In the case when the surface S admits only two poles and conical singularities of even degree, we can still define a spin structure in the following way: the two poles have opposite residues, hence by cutting the poles and gluing together the corresponding cylinders, we get a standard translation surface $\widetilde{S}$ (it is in fact a closing of S) in a stratum where the spin structure is well defined. The different choices for this operation give the same connected components. Hence we define the parity of spin structure of S to be the parity of spin structure of the corresponding standard translation surface $\widetilde{S}$.

Remark 5.7. Note that the formula for spin structure will not be invariant if we only consider basis for the homology of S.

6. Higher genus case: minimal stratum

A minimal stratum correspond to the case where there is only one conical singularities (and possibly several poles). As in [11] and in [12], we first describe the connected components of minimal strata. The idea is similar: show that each such strata is obtained by bubbling g cylinders and compute the connected components in this case.

The first step is to find a surface obtained by bubbling a handle. In [11] and in [12] is used a rather combinatorial argument. A similar approach is possible in our case by using the infinite zippered rectangle construction, but this is quite technical. Another possibility is to reduce the problem to the genus one case for which it was proven in Section 4 that any minimal stratum contains a surface obtained by bubbling a handle.

Proposition 6.1. *Let $\mathcal{C}$ be a connected component of the stratum $\mathcal{H}(n, -p1, \dots, -p_s)$. We assume that the genus g is nonzero. Then, there exists a flat surface in $\mathcal{C}$ which is obtained by bubbling a handle from a genus $g-1$ flat surface.*

Proof. We start from a surface in $\mathcal{C}$ obtained by the infinite zippered rectangle construction. It is defined by a combinatorial data and a continous parameter $\zeta \in \mathbb{C}^n$, with $n = 2g + s - 1$.

Each ζ_i defines a closed geodesic path γ_i joining the conical singularity to itself. The intersection number between any two such path is 0 or ± 1. We claim that there is a pair γ_i, γ_j whose intersection number is one. Indeed, the genus is higher than zero and $\{\gamma_1, \dots, \gamma_n\}$ generates the whole homology space $H_1(S, \mathbb{Z})$ since the complement is a union of punctured disks.

Now we shrink ζ_i, ζ_j until they are very small compared to all the other parameters. Then, we observe that a neighborhood of γ_i, γ_j is isometric to the complement of a neighborhood of a pole for a surface in $\mathcal{H}(n, -n)$. Then, deforming suitably the surface, using Proposition 4.4, one obtains the desired result. □

We recall the notation introduced in Section 3.2. Let $\mathcal{C}$ is a connected component of a minimal stratum $\mathcal{H}(n, -p_1, \dots, -p_s)$. Let $s \in \{1, \dots, n+1\}$. The set $\mathcal{C} \oplus s$ is the connected component of the stratum $\mathcal{H}(n+2, -p_1, \dots, -p_s)$ obtained by bubbling a handle after breaking the singularity of order n into two singularities of order $(s-1)$ and $(n+1-s)$.

The proposition that follows uses roughtly the same arguments as in [11] and [12]. The only difference is the case when n is odd, which does not occur for Abelian or quadratic differentials.

Proposition 6.2. *Let $\mathcal{H}(n, -p_1, \dots, -p_s)$ be a stratum of meromorphic differentials genus $g \geq 2$ surfaces, and denote by $\mathcal{C}_0$ the unique component of $\mathcal{H}(n-2g, -p_1, \dots, -p_s)$. The following holds:*

- *If n is odd, the stratum $\mathcal{H}(n, -p_1, \dots, -p_s)$ is connected.*
- *If n is even, the stratum $\mathcal{H}(n, -p_1, \dots, -p_s)$ has at most three connected components which are in the following list:*

– $\mathcal{C}_0 \oplus (\frac{n-2g}{2}+1) \oplus (\frac{n-2g}{2}+2) \oplus \cdots \oplus (\frac{n-2g}{2}+g)$
– $\mathcal{C}_0 \oplus 1 \oplus \cdots \oplus 1 \oplus 1$
– $\mathcal{C}_0 \oplus 1 \oplus \cdots \oplus 1 \oplus 2$

Proof. Let $\mathcal{C}$ be a connected component of $\mathcal{H}(n, -p_1, \dots, -p_s)$. By proposition 6.1, there exists integers $s_1, \dots, s_g$, such that:

$$\mathcal{C} = \mathcal{C}_0 \oplus s_1 \oplus \cdots \oplus s_g$$

and for each $i \in \{1, \dots, g\}$, $1 \leq s_i \leq n - 2g - 2 + 2i + 1$, since at Step i, the handle corresponding to s_i is bubbled on a zero of degree $n - 2g + 2(i-1)$.

We assume for simplicity that $g = 2$, and $(s_1, s_2) \neq (\frac{n-2g}{2}+1, \frac{n-2g}{2}+2)$. Using operations (1) and (3) of Lemma 3.1, one can assume that $1 \leq s_1 \leq s_2 \leq s_1 + 1$. Then, if $1 \neq s_1$, using operations (1), (2), (3) and (1) (in this order), we have $\mathcal{C}_0 \oplus s_1 \oplus s_2 = \mathcal{C}_0 \oplus (s_1 - 1) \oplus (s_2 - 1)$. Repeating the same sequence of operations, we see that $\mathcal{C}$ is one of the following:

- $\mathcal{C}_0 \oplus (\frac{n-2g}{2}+1) \oplus (\frac{n-2g}{2}+2)$
- $\mathcal{C}_0 \oplus 1 \oplus 1$
- $\mathcal{C}_0 \oplus 1 \oplus 2$

If n is odd, then the first case doesn't appear. By operation (4) of Lemma 3.1, we have

$$\mathcal{C}_0 \oplus s_1 \oplus s_2 = \mathcal{C}_0 \oplus s_1 \oplus ((n - 2g + 2) + 2 - s_2)$$

so we can assume that s_1 and s_2 are of the same parity. Then, using the previous argument, we have:

$$\mathcal{C} = \mathcal{C}_0 \oplus 1 \oplus 1$$

The case $g > 2$ easily follows. □

The above proposition uses purely local constructions in a neighborhood of a singularity. The next proposition explains why the existence of suitable poles (at infinity) will "kill" some components.

Proposition 6.3. *Let $\mathcal{H}(n, -p_1, \dots, -p_s)$ be a stratum of meromorphic differentials genus $g \geq 2$ surfaces with n even and $s \geq 2$, and denote by $\mathcal{C}_0$ the unique component of $\mathcal{H}(n-2g, -p_1, \dots, -p_s)$. The following holds:*

(1) *If there is a odd degree pole and $\sum_i p_i > 2$, then:*

$$\mathcal{C}_0 \oplus 1 \oplus \cdots \oplus 1 = \mathcal{C}_0 \oplus 1 \oplus \cdots \oplus 1 \oplus 2$$

(2) *If $s > 2$ or $p_1 \neq p_2$, then:*

$$\mathcal{C}_0 \oplus \left(\frac{n-2g}{2}+1\right) \oplus \cdots \oplus \left(\frac{n-2g}{2}+g\right) = \mathcal{C}_0 \oplus 1 \oplus \cdots \oplus 1 \oplus s$$

for some $s \in \{1,2\}$.

Proof. Case (1).
Note that $s \geq 2$ implies that we necessarily have $\sum_i p_i \geq 2$. From Proposition 4.4, $\mathcal{C}_0 \oplus 2 = \mathcal{C}_0 \oplus k$ if and only if $\gcd(k, p_1, \dots, p_s) = \gcd(2, p_1, \dots, p_s)$. So, if there is an odd degree pole, $\gcd(2, p_1, \dots, p_s) = 1 = \gcd(1, p_1, \dots, p_s)$, hence

$$\mathcal{C}_0 \oplus 1 \cdots \oplus 1 = \mathcal{C}_0 \oplus 2 \oplus 1 \cdots \oplus 1 = \mathcal{C}_0 \oplus 1 \cdots \oplus 1 \oplus 2,$$

which concludes the proof.

Case (2).
As before, we use the classification in genus one. Since $n - 2g - \sum_i p_i = -2$, we have $\frac{n-2g}{2} + 1 = \frac{\sum_i p_i}{2}$. If $s > 2$ or $p_1 \neq p_2$, then there exists $i \in \{1, \dots, s\}$ such that $\frac{n-2g}{2} + 1 > p_i$, so $\gcd(\frac{n-2g}{2} + 1, p_1, \dots, p_s) < \frac{n-2g}{2} + 1$, hence there exists $k < \frac{n-2g}{2} + 1$ such that $\mathcal{C}_0 \oplus (\frac{n-2g}{2} + 1) = \mathcal{C}_0 \oplus k$. So we have

$$\mathcal{C}_0 \oplus (\frac{n-2g}{2} + 1) \oplus (\frac{n-2g}{2} + 2) \oplus \cdots = \mathcal{C}_0 \oplus k \oplus (\frac{n-2g}{2} + 2) \oplus \dots$$

Then, as in the proof of Proposition 6.2,

$$\mathcal{C}_0 \oplus k \oplus (\frac{n-2g}{2} + 2) \cdots \oplus (\frac{n-2g}{2} + g) = \mathcal{C}_0 \oplus 1 \oplus \cdots \oplus 1 \oplus s$$

for some $s \in \{1,2\}$. □

Putting together the last two propositions and the invariants, we have the following theorem.

Theorem 6.4. *Let* $\mathcal{H} = \mathcal{H}(n, -p_1, \dots, -p_s)$ *be a minimal stratum of meromorphic differentials on genus* $g \geq 2$ *surfaces. We have:*

1. *If* n *is even and* $s = 1$, *then* $\mathcal{H}$ *has two connected components if* $g = 2$ *and* $p_s = 2$, *three otherwise.*
2. *If* $\mathcal{H} = \mathcal{H}(n, -p, -p)$, *with* p *even, then* $\mathcal{H}$ *has three connected components.*
3. *If* $\mathcal{H} = \mathcal{H}(n, -1, -1)$, *then* $\mathcal{H}$ *has three connected components for* $g > 2$, *two otherwise.*
4. *If* $\mathcal{H} = \mathcal{H}(n, -p, -p)$, *with* $p \neq 1$ *odd, then* $\mathcal{H}$ *has two connected components.*
5. *If all poles are even and we are not in the previous case, then* $\mathcal{H}$ *has two connected components.*
6. *In the remaining cases,* $\mathcal{H}$ *is connected.*

Proof. From Proposition 6.2, when n is odd, which is part of Case (6), $\mathcal{H}$ is connected. So we can assume that n is even. Let $\mathcal{C}$ be a connected component of $\mathcal{H}$. Let $\mathcal{C}_0$ be the (connected) genus 0 stratum $\mathcal{H}(n -$

$2g, -p_1, \dots, -p_s)$. From Proposition 6.2, we have one of the three following possibilities.

a) $\mathcal{C} = \mathcal{C}_0 \oplus (\frac{n-2g}{2}+1) \oplus (\frac{n-2g}{2}+2) \oplus \cdots \oplus \frac{n}{2}$
b) $\mathcal{C} = \mathcal{C}_0 \oplus 1 \oplus \cdots \oplus 1 \oplus 1$
c) $\mathcal{C} = \mathcal{C}_0 \oplus 1 \oplus \cdots \oplus 1 \oplus 2$

When $\mathcal{H} = \mathcal{H}(n, -p)$ or $\mathcal{H} = \mathcal{H}(n, -p, -p)$, it is easy to see that case *a*) corresponds to a hyperelliptic connected component, while case *b*) do not, and neither *c*) (except for the case $n-2g=0$ and $g=2$, where *a*) and *c*) are the same).

When all degree of zeroes (and poles) are even, then Lemma 11 in [11] shows that cases b) and c) corresponds to different spin structure, so are a different connected component. This is also true for $\mathcal{H}(n, -1, -1)$ by Section 5.3.

The arguments of the two previous paragraphs proves the result for Cases (1), (2) and (3). Remark that Riemann-Roch theorem implies that $n - 2g = \sum_i p_i - 2$.

For Case (4), Proposition 6.3 shows that there are at most two connected components. Since $n - 2g = 2p - 2 > 0$, Case *a*) corresponds to a hyperelliptic component while *b*)and *c*) do not correspond to a hyperelliptic component. So there are at least two components. Since there are odd degree poles, *b*) and *c*) correspond to the same component by Proposition 6.3. So there are two components.

For Case (5), Proposition 6.3 shows that *a*) is in the same connected component as *b*) or *c*), while Lemma 11 in [11] shows that *b*) and *c*) have different spin structures.

For Case (6), with n is even: this corresponds to having at least one odd pole, and either at least three poles or two poles of different degree. Then a direct application of Proposition 6.3 shows that *a*), *b*) and *c*) are the same connected component.

This concludes the proof.

□

7. Higher genus case: nonminimal strata

The remaining part of the paper uses similar arguments as in Sections 5.2–5.4 in [11]. We quickly recall the three main steps.

- Each stratum is adjacent to a minimal stratum, and we can bound the number of connected components of a stratum by the number of connected components of the corresponding minimal one.

- We construct paths in suitable strata with two conical singularities that join the different connected components of the minimal stratum.
- We deduce from the previous arguments upper bounds on the number of connected component of a stratum, lower bounds are given by the topological invariants.

The following proposition is analogous to Corollary 4 in [11]. It is proven there by constructing surfaces with a one cylinder decomposition. Such surfaces never exists in our case, we use the infinite zippered rectangle construction instead.

Proposition 7.1. *Any connected component of a stratum of meromorphic differentials is adjacent to the minimal stratum obtained by collapsing together all the zeroes.*

Proof. Let S be in a stratum $\mathcal{H}$ of meromorphic differentials. We prove the result by induction on the number of conical singularities of S. We can assume that S is obained by the previous construction. By connexity of S, there is a $D^{\pm}$ component or a $C^{\pm}$ component that contains two different conical singularities on its boundary, hence, there is a parameter ζ_i whose corresponding segment on that component joins two different conical singularities. The segment is on the boundary of two components. Assume for instance, that it is a D^+ and a C^- component. Now we just need to check that the surface obtained by shrinking ζ_i to zero is nondegenerate. Hence it will correspond to an element in a stratum with one less conical singularity. The set D'^+ obtained by shrinking ζ_i to zero from D^+ is still a domain as defined in Section 3.3.2. The set C'^- obtained by shrinking ζ_i to zero from C^- is also a domain as defined in Section 3.3.2 except if we have $C^- = C^-(\zeta_i)$. But in this case, since the two vertical lines of C^- are identified together, the two endpoints of the segment defined by ζ_i are necessarily the same singularity, contradicting the hypothesis.

So, in any case, we obtain a surface S' with fewer conical singularities. $\square$

The following proposition is analogous to Corollary 2 in [11], and is the first step of the proof described in the beginning of this section. The proof of Kontsevich and Zorich uses a deformation theory argument. We propose a proof that uses only flat geometry.

Proposition 7.2. *The number of connected component of a stratum is smaller than or equal to the number of connected component of the corresponding minimal stratum.*

Proof. From the previous proposition, any connected component of a stratum $\mathcal{H} = \mathcal{H}(k_1, \dots, k_r, -p_1, \dots, -p_s)$ is adjacent to a minimal stratum $\mathcal{H}^{min} = \mathcal{H}(k_1 +, \dots + k_r, -p_1, \dots, -p_s)$ by collapsing zeroes. It is enough to show that if (S_n), (S'_n) are two sequences in $\mathcal{H}$ that converge to a surface $S \in \mathcal{H}^{min}$, then S_n and S'_n are in the same connected component of $\mathcal{H}$ for n large enough.

By definition of the topology on the moduli space of meromorphic differentials , for n large enough, the conical singularities of S_n (resp. S'_n) are all in a small disk D_n (resp. D'_n) which is embedded in the surface S_n (resp. S'_n), and whose boundary is a covering of a metric circle.

Note that D_n and D'_n can be chosen arbitrarily small if n is large enough, and we can assume that they have isometric boundaries. Replacing D_n by a disk with a single singularity, one obtain a translation surface $\widetilde{S}_n$ which is very near to S, hence in the same connected component, and similarly for S'_n.

Now we want to deform D_n to obtain D'_n. It is obtained in the following way: D_n can be seen as a subset of a genus zero translation surface S_1 in the stratum $\mathcal{H}(k_1, \dots, k_r, -2 - \sum_{i=1}^r k_i)$: we just "glue" a neighborhood of a pole to the boundary of the disk D_n. We proceed similarly with the disk D'_n and obtain a translation surface S_2 in the same stratum as S_1. This stratum is connected since the genus is zero. Hence we deduce a continuous transformation that deform D_n to D'_n.

From the last two paragraph, we easily deduce a continuous path from S_n to S'_n, which proves the proposition. □

The following proposition is the second step of the proof. It is the analogous of Proposition 5 and Proposition 6 in [11]. Our proof is also valid for the Abelian case, and gives an interesting alternate proof.

Proposition 7.3.
1. *Let $\mathcal{H} = \mathcal{H}(n, -p_1, \dots, -p_s)$ be a genus $g \geq 2$ minimal stratum whose poles are all even or the pair $(-1, -1)$. For any n_1, n_2 odd such that $n_1 + n_2 = n$, there is a path $\gamma(t) \in \overline{\mathcal{H}(n_1, n_2, -p_1, \dots, -p_s)}$ such that $\gamma(0), \gamma(1) \in \mathcal{H}$ and have different parities of spin structures.*
2. *Let $\mathcal{H} = \mathcal{H}(n, -p_1, \dots, -p_s)$ be a genus $g \geq 2$ minimal stratum that contains a hyperelliptic connected component. For any $n_1 \neq n_2$ such that $n_1 + n_2 = n$, there is a path $\gamma(t) \in \overline{\mathcal{H}(n_1, n_2, -p_1, \dots, -p_s)}$ such that $\gamma(0)$ is in a hyperelliptic component of $\mathcal{H}$ and $\gamma(1)$ is in a nonhyperelliptic component of $\mathcal{H}$.*

Proof. Case (1)
Let $\mathcal{C}_0 = \mathcal{H}(n - 2g, -p_1, \dots, -p_s)$. The connected components of $\mathcal{H}$

given by $\mathcal{C}_0 \oplus 1 \cdots \oplus 1 \oplus 1$ and $\mathcal{C}_0 \oplus 1 \cdots \oplus 1 \oplus 2$ have different parities of spin structures. We can rewrite these components as $\mathcal{C} \oplus 1$ and $\mathcal{C} \oplus 2$, where $\mathcal{C} = \mathcal{C}_0 \oplus 1 \cdots \oplus 1$.

Fix $S_{g-1} \in \mathcal{C}$. For a surface $S_1 \in \mathcal{H}(n, -n)$, one can get a surface S in $\mathcal{H}(n, -p_1, \dots, -p_s)$ by the following surgery:

- Cut S_{g-1} along a small metric circle that turns around the singularity of degree $n-2$, and remove the disk bounded by this circle
- Cut S_1 along a large circle that turns around the pole of order n, and rescale S_1 such that this circle is isometric to the previous one. Remove the neighborhood of the pole of order n bounded by this circle.
- Glue the two remaning surfaces along these circle, to obtain a surface $S \in \mathcal{H}(n, -p_1, \dots, -p_s)$.

All choices in previous construction lead to the same connected component of $\mathcal{H}(n, -p_1, \dots, -p_s)$, once S_{g-1}, S_1 are fixed. Similarly, we can do the same starting from a surface in $S_1 \in \mathcal{H}(n_1, n_2, -n)$ and get a surface in $\mathcal{H}(n_1, n_2, -p_1, \dots, -p_s)$.

Now we start from a surface $S_{1,1} \in \mathcal{H}(n, -n)$ obtained by bubbling a handle with angle 2π, *i.e.* $S_{1,1} \in \mathcal{H}(n-2, -n) \oplus 1$. The rotation number of this surface is $\gcd(1, n) = 1$. Breaking up the singularity into two singularities of order n_1, n_2, the rotation number is still 1. Similarly, start from $S_{1,2} \in \mathcal{H}(n-2, -n) \oplus 2$. Its rotation number is $\gcd(2, n) = 2$. Breaking up the singularity into two singularities of order n_1, n_2, the rotation number becomes $\gcd(2, n_1, n_2) = 1$ since n_1, n_2 are odd. Hence there is a path in $\overline{\mathcal{H}(n_1, n_2, -n)}$ that joins $S_{1,1} \in \mathcal{H}(n-2, -n) \oplus 1$ to $S_{1,2} \in \mathcal{H}(n-2, -n) \oplus 2$. From this path, we deduce a path in $\overline{\mathcal{H}(n_1, n_2, -p_1, \dots, -p_s)}$ that joins $\mathcal{C} \oplus 1$ to $\mathcal{C} \oplus 2$. So Part (1) of the proposition is proven.

Case (2)
The proof is similar as the previous one: the hyperelliptic component of $\mathcal{H}(n, -p_1, \dots, -p_s)$ is of the kind $\mathcal{C} \oplus \frac{n}{2}$, for some component $\mathcal{C}$. Any component of the kind $\mathcal{C} \oplus k$, with $k \neq \frac{n}{2}$ is nonhyperelliptic. As before, we reduce use genus one strata. A surface in $\mathcal{H}(n-2, -n) \oplus \frac{n}{2}$ is of rotation number $\gcd(\frac{n}{2}, n) = \frac{n}{2}$. Breaking up the singularity of degree n into two singularities of degree n_1, n_2, one obtain surface in $\mathcal{H}(n_1, n_2, -n)$ of rotation number $\gcd(\frac{n}{2}, n_1, n_2)$. Since $n_1 + n_2 = n$ and $n_1 \neq n_2$, this rotation number is not $\frac{n}{2}$, but some integer $k \in \{1, \dots, \frac{n}{2} - 1\}$. Hence there is a path in $\overline{\mathcal{H}(n_1, n_2, -n)}$ that joins $\mathcal{H}(n-$

$2,-n) \oplus \frac{n}{2}$ to $\mathcal{H}(n-2,-n) \oplus k$. From this, we deduce the required path in $\overline{\mathcal{H}(n_1,n_2,-p_1,\dots,-p_s)}$. □

Now we have all the intermediary results to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\mathcal{H} = \mathcal{H}(n_1,\dots,n_r,-p_1,\dots,-p_s)$ be a stratum of genus $g \geq 2$ surfaces. Denote by $\mathcal{H}_{min}$ the minimal stratum obtained by collapsing all zeroes. Recall that by Proposition 7.2, the number of connected components of $\mathcal{H}$ is smaller than, or equal to the number of connected components of $\mathcal{H}_{min}$.

If $\sum_i p_i$ is odd, then the minimal stratum is connected and therefore the stratum is connected. So we can assume that $\sum_i p_i$ is even.

Assume that $\sum p_i > 2$ or $g > 2$. From Theorem 6.4, $\mathcal{H}_{min}$, hence $\mathcal{H}$ has at most three components.

We fix some vocabulary: we say that the set of degree of zeroes (resp. poles) is of *hyperelliptic type* if this set is $\{n,n\}$ or $\{2n\}$ (resp. $\{-p,-p\}$ or $\{-2p\}$), *i.e.* it is the set of degree of zeroes or poles of a hyperelliptic component. Note that the set of degree of poles are of hyperelliptic type if and only if the corresponding minimal stratum contains a hyperelliptic connected component. We will also say that the set of degree of poles is of *even type* if they are all even or if they are $\{-1,-1\}$. This means that the underlying minimal stratum has two nonhyperelliptic components distinghished by the parity of spin structure.

- If the stratum is $\mathcal{H}(n,n,-2p)$ or $\mathcal{H}(n,n,-p,-p)$. There is a hyperelliptic connected component. The corresponding minimal stratum $\mathcal{H}(2n,*)$ has one hyperelliptic component and at least one nonhypereliptic component. It is easy to see that breaking up the singularity of degree $2n$ into two singularities of degree n, from a nonhyperelliptic translation surface gives a surface in a nonhyperelliptic connected component. So, the stratum $\mathcal{H}(n,n,*)$ has one hyperelliptic connected component and at least one nonhyperelliptic connected component.
- If the set of degrees of poles and zeroes is of even type, we know from Theorem 6.4 that the minimal stratum has two nonhyperelliptic components (and possibly one hyperelliptic). Breaking up the singularity into even degree singularities preserves the spin structure, which therefore gives at least two nonhyperelliptic components in the stratum.

From the above description, we obtain lower bounds on the number of connected component. In particular, we see that if the degree of zeroes and poles are both of hyperelliptic and even type, $\mathcal{H}$ as at least, so

exactly, three connected components. Also, if the the set of degrees of zeroes and poles are of hyperelliptic or even type, $\mathcal{H}$ as at least two connected components.

Now we give upper bounds.

(1) Assume that the poles are of hyperelliptic and even type, *i.e.* the minimal stratum has three connected components. Denote respectively by $\mathcal{C}^{hyp}, \mathcal{C}^{odd}$ and $\mathcal{C}^{even}$ the connected components of $\mathcal{H}$ that are adjacent respectively to the three connected components of $\mathcal{H}_{min}$, $\mathcal{H}_{min}^{hyp}, \mathcal{H}_{min}^{odd}$ and $\mathcal{H}_{min}^{even}$. For any $j \in \{1, \dots, r\}$, the stratum $\mathcal{H}(n_j, \sum_{i\neq j} n_i, -p_1, \dots, -p_s)$ is adjacent to $\mathcal{H}_{min}$.
   - If the zeroes are not of hyperelliptic type, we can choose, n_i so that $n_i \neq \sum_{i\neq j} n_i$, and by Proposition 7.3 there is a path in $\mathcal{H}(n_j, \sum_{i\neq j} n_i, -p_1, \dots, -p_s)$ joining the hyperelliptic component of $\mathcal{H}_{min}$ to a nonhyperelliptic connected component. Breaking up the singularity of order $\sum_{i\neq j}^{r} n_i$ along this path into singularities of order $(n_i)_{i\neq j}$, we obtain a path in $\mathcal{H}$ that joins a neighborhood of $\mathcal{H}_{min}^{hyp}$ to a neighborhood of a nonhyperelliptic component of $\mathcal{H}_{min}$. Hence, we necessarily have $\mathcal{C}^{hyp} = \mathcal{C}^{odd}$ or $\mathcal{C}^{hyp} = \mathcal{C}^{even}$.
   - If the zeroes are not even, we conclude similarly that $\mathcal{C}^{odd} = \mathcal{C}^{even}$
   - Note that if the zeroes are neither of hyperelliptic type nor of even type, then $\mathcal{C}^{even} = \mathcal{C}^{odd} = \mathcal{C}^{hyp}$, so there is only one component for $\mathcal{H}$.

(2) Assume that the poles are of hyperelliptic type but not of even type. The minimal stratum has two connected components, so there are at most two connected components for $\mathcal{H}$. If the zeroes are of hyperelliptic type, we have already seen that there are two components.

   Assume the zeroes are not of hyperelliptic type. Denote respectively by $\mathcal{C}^{hyp}, \mathcal{C}^{nonhyp}$ the connected components of $\mathcal{H}$ that are adjacent respectively to the hyperelliptic and the nonhyperelliptic component of $\mathcal{H}_{min}$. By the same argument as in (1), using Proposition 7.3 we have $\mathcal{C}^{hyp} = \mathcal{C}^{nonhyp}$, so $\mathcal{H}$ is connected.

(3) Assume that the poles are of even type but not of hyperelliptic type. The minimal stratum has two connected components distinguished by the parity of spin structure. So there are at most two components for $\mathcal{H}$. If the zeroes are of even type, there are exactly two connected component for $\mathcal{H}$, that are distinguished by the parity of spin structure.

If the zeroes are not of even type, denote respectively by $\mathcal{C}^{odd}, \mathcal{C}^{even}$ the connected components of $\mathcal{H}$ that are adjacent respectively to the two components of $\mathcal{H}_{min}$. By the same argument as in (1), using Proposition 7.3 we have $\mathcal{C}^{odd} = \mathcal{C}^{even}$.

(4) Assume that the poles are neither of hyperelliptic nor of even type, then the minimal stratum is connected, so $\mathcal{H}$ is connected.

It remains to prove the theorem when $g = 2$ and $\sum_i p_i = 2$. The minimal stratum has two connected components. In this case, it is equivalent to say that the zeroes are of hyperelliptic type or to say that they are of even type. If $\mathcal{H} = \mathcal{H}(2, 2, *)$ or $\mathcal{H}(4, *)$, the stratum has at least two components, so exactly two. Otherwise, the stratum is adjacent to $\mathcal{H}(3, 1, *)$, which connects $\mathcal{H}_{min}^{odd}$ to $\mathcal{H}_{min}^{even}$, hence $\mathcal{H}$ is connected. □

Appendix A. Negative results for meromorphic differentials

In this section, we quickly give some examples to show that many well known results for the dynamics on translation surfaces are false in the case of translation surfaces with poles.

A.1. **Dynamics of the geodesic flow.** On a standard translation surface, the geodesic flow is uniquely ergodic for almost any directions. From the result of Proposition 3.6, for almost any direction on a translation surface with poles, all infinite orbits for the geodesic flow converge to a pole.

A.2. **Cylinders and closed geodesics.** On a standard translation surface, always exists infinitely many closed geodesics (hence cylinders). For the case of translation surface with poles, one can consider the following example. Take the plane $\mathbb{C}$ and remove the inside of a square, and glue together by translation the corresponding opposite sides. One gets a surface in $\mathcal{H}(-2, 2)$. It is easy to see that there are exactly two saddle connections joining the conical singularity to itself and no closed geodesic. A similar example in $\mathcal{H}(-2, 1, 1)$ obtained by removing a regular hexagon gives an example with not a single saddle connection joining a conical singularity to itself.

A.3. $SL_2(\mathbb{R})$ **action.** The $SL_2(\mathbb{R})$ action on the strata of the moduli space of Abelian differentials is ergodic. It is not the case for the moduli space of meromorphic differentials if we consider the (infinite) volume form defined by the flat local coordinates. Indeed, consider the stratum $\mathcal{H}(-2, 2)$, which is connected. Consider the set of surfaces obtained with the infinite zippered rectangle construction, by gluing

together the set $D^+(z_1, z_2)$ and the set $D^-(z_2, z_1)$. It is easy to see that if $Im(z_2) < 0 < Im(z_1)$, there are no cylinders on the surface while if $Im(z_2) > 0 > Im(z_1)$, there is a cylinder on the surface. These two cases form two nointersecting open subsets of $\mathcal{H}(-2, 2)$. Considering $SL_2(\mathbb{R})$ orbits, we obtain two disjoints $SL_2(\mathbb{R})$-invariants open subsets of a connected stratum.

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